1. The examination contains 4 problems. You have 75 minutes for 40 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [10 points]
   (a) [6 points] Let the alphabet \( \Sigma = \{a, b\} \). A string \( x \in \Sigma^* \) is called a palindrome if \( x = x^r \) (\( x \) reads the same backward as forward); for example, the string \( abaaaba \) is a palindrome.

   Consider the following binary relation \( R \) on \( \Sigma^* \): For all \( x, y \in \Sigma^* \), \( xRy \) if and only if \( |x| = |y| \) and \( xy^r \) is a palindrome. Note: \( xy^r \) is \( x(y^r) \), not \( (xy)^r \).

   i. [5 points] Show that \( R \) is an equivalence relation on \( \Sigma^* \).

   ii. [1 point] For an arbitrary \( x \in \Sigma^* \), describe the equivalence class \([x]_R\) (the equivalence class of \( R \) containing \( x \)). Justify your answer.
(b) [4 points] Let $L$ be a language over an alphabet $\Sigma$ such that $L \neq \emptyset$, $L \neq \{\epsilon\}$, and $L^2 = L$. Prove that:
   
   i. [2 points] $\epsilon \in L$, and
   
   ii. [2 points] $L$ is not a finite language (that is, $|L|$ is not finite).
2. [10 points]

(a) [5 points] Give the state-transition diagram of a deterministic finite automaton that accepts the following language:

\[ \{x \in \{a, b\}^* \mid |\#_a(u) - \#_b(u)| \leq 2 \text{ for every prefix } u \text{ of } x \}, \]

where \#_c(u) denotes the number of occurrences of the symbol \( c \) in the string \( u \). Give a brief interpretation of the states in your construction.
(b) [5 points] Let \( \Sigma = \{a, b\} \), \( M = (Q, \Sigma, \delta, q_0, A) \) be a deterministic finite automaton, and

\[
E_M = \{ x \mid x \in L(M), \text{ and } |x| \text{ is even} \}.
\]

Show that \( E_M \) is regular by giving explicitly the 5-tuple definition of a deterministic finite automaton accepting \( E_M \). Give a brief interpretation of the states in your construction.
3. [10 points] Let $\Sigma$ be the alphabet of 3 symbols: $\Sigma = \{a, b, c\}$.

(a) [5 points] Consider the following language:

$$L_1 = \{w \in \Sigma^* \mid \text{there exists a symbol of } \Sigma \text{ not appearing in } w\}.$$ 

Give the formal 5-tuple definition and the state-transition diagram of a nondeterministic finite automaton with at most 5-states (with or without $\epsilon$-transitions) that accepts $L_1$. Notes: A brief and precise interpretation of the states of your machine is required, and no credit will be given to finite automaton with more than 5 states.
(b) [5 points] Consider the following language:

\[ L_2 = \{ w \in \Sigma^* \mid w \text{ has a substring of length 3 containing each of the symbols of } \Sigma \}. \]

Give the state-transition diagram of a nondeterministic finite automaton (with or without \( \epsilon \)-transitions) with at most 8 states that accepts \( L_2 \). Notes: A brief and precise interpretation of the states of your machine is required, and no credit will be given to finite automaton with more than 8 states.
4. [10 points] Consider the following nondeterministic finite automaton $M_1$ with $\epsilon$-transitions:

![Diagram](image)

Figure 1: A nondeterministic finite automaton with $\epsilon$-transitions.

(a) [5 points] Construct the state-transition diagram of a nondeterministic finite automaton without $\epsilon$-transitions $M_2$ that is equivalent to $M_1$. Show all the intermediate steps – including the computation of all $\epsilon$-closures.
(b) [5 points] Construct, by using the Subset Construction, the state-transition diagram of a deterministic finite automaton $M_3$ (with its inaccessible states removed) that is equivalent to $M_2$. 