1. The examination contains 3 problems. You have 50 minutes for 30 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [6 points] Let $L$ be a language over an alphabet $\Sigma$ such that $L \neq \emptyset$, $L \neq \{\epsilon\}$, and $L^2 = L$. Prove that:
   
   (a) [3 points] $\epsilon \in L$, and

   (b) [3 points] $L$ is not a finite language (that is, $|L|$ is not finite).
2. [10 points]

(a) [5 points] Give the state-transition diagram of a deterministic finite automaton that accepts the following language:

\[ \{ x \in \{a, b\}^* \mid |\#_a(u) - \#_b(u)| \leq 2 \text{ for every prefix } u \text{ of } x \}, \]

where \( \#_c(u) \) denotes the number of occurrences of the symbol \( c \) in the string \( u \). Give a brief interpretation of the states in your construction.
(b) [5 points] Let $\Sigma = \{a, b\}$, $M = (Q, \Sigma, \delta, q_0, A)$ be a deterministic finite automaton, and

$$E_M = \{ x \mid x \in L(M), \text{ and } |x| \text{ is even} \}.$$ 

Show that $E_M$ is regular by giving explicitly the 5-tuple definition of a deterministic finite automaton accepting $E_M$. Give a brief interpretation of the states in your construction.
3. [14 points] Let $\Sigma$ be the alphabet of 3 symbols: $\Sigma = \{a, b, c\}$. Consider the following language over $\Sigma$:

$$L = \{w \in \Sigma^* \mid \text{there exists at least one symbol of } \Sigma \text{ not appearing in } w\}.$$ 

For examples: $\epsilon, a, bb, c, acaa \in L$, but $bbac, aaaccebb \notin L$.

(a) [7 points] Give the formal 5-tuple definition and the state-transition diagram of a nondeterministic finite automaton with at most 5 states (with or without $\epsilon$-transitions) that accepts $L$. Notes: A brief and precise interpretation of the states of your machine is required, and no credit will be given to finite automaton with more than 5 states.
(b) [2 points] Give a regular expression that denotes/represents $L$. Annotate your regular expression with explanation.

(c) [5 points] Prove that every deterministic finite automaton accepting $L$ must have at least $2^3$ states.