1. The “closed-book/notes” examination contains 3 problems. You have 110 minutes for 45 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [15 points]
   
   (a) [4 points] Assume that $\Sigma$ is an alphabet, and $A \subseteq \Sigma^*$ is an infinite language. Prove or disprove that there exists a non-regular language $B \subseteq \Sigma^*$ such that $B \subseteq A$. 
(b) [6 points] Let $\Sigma = \{0, 1\}$, and suppose that $R \subseteq \Sigma^*$ is an arbitrarily chosen regular language. Define a language over $\Sigma$ as follows:

$$L = \{ w \in \Sigma^* \mid \text{there exist strings } u, v \in \Sigma^* \text{ such that } uwv \in R \}.$$ 

In other words, the language $L$ contains every string that is a substring of a string in $R$. Prove or disprove that $L$ is a regular language.
(c) [5 points] Let $\Sigma = \{0, 1\}$, and define a language:

$$L = \{u0v \mid u, v \in \Sigma^* \text{ and } |u| = |v|\}.$$ 

In other words, $L$ is the language of all binary strings of odd length whose middle symbol is 0. Apply the pumping lemma directly on $L$ to prove its non-regularity.
2. [15 points]

(a) [4 points] Construct a context-free grammar that generates the following language:

\[ L = \{ w \in \{0, 1, 2\}^* \mid \#_0(w) = \#_1(w) + \#_2(w) \} \].

Notes: (1) For two strings \( u \) and \( v \), \( \#_u(v) \) denotes the number of occurrences of \( u \) as a substring of \( v \), and (2) No credits will be given for construction without brief and precise interpretations of the variables of your grammar.
(b) [6 points] Suppose that $R$ is a regular language. Prove or disprove the context-freedom of the following language:

$$L = \{ww^r \mid w \in R\}.$$  

Note: For a string $u$, $u^r$ denotes the reversal of $u$. 


(c) [5 points] Show that the following language is not context-free by using pumping lemma:

\[ L = \{(0^k1^k)^k \mid k \geq 1\} \].
3. [15 points]

(a) [4 points] Design a Turing machine that decides the language:

$$L = \{0^m1^n \mid 1 \leq m \leq n\}.$$ 

Give a complete description of the transition function and brief and precise interpretations of the states and transitions of your construction.
(b) [6 points] Let $\Sigma$ be an alphabet, and assume that $A, B \subseteq \Sigma^*$ are Turing-recognizable languages such that both $A \cap B$ and $A \cup B$ are decidable. Prove that $A$ is decidable.
(c) [5 points] Let $\Sigma = \{0, 1\}$, and define a language $A \subseteq \Sigma^*$ as:

$$A = \{\langle M \rangle \mid \langle M \rangle \text{ is a deterministic Turing machine that halts on at least one input string} \}.$$ 

Prove that $A$ is Turing-recognizable.